Final Project

Review II

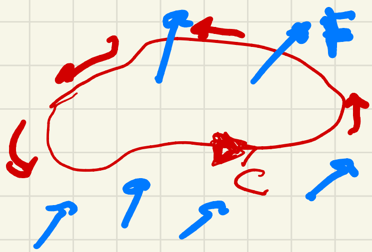
Problem 5

Problem 5. The meaning of the operation curl on vector fields

A. Circulation of the flow around a closed curve

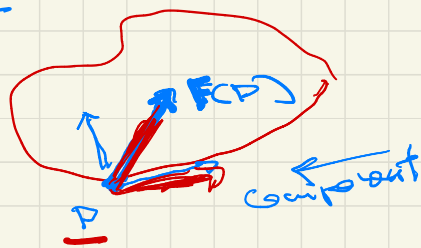
We will see that curl is a measure of circulation.

Consider a flow of some quantity Q with velocity vector F (you can also think the F is the force vector field that makes pieces of Q flow!).



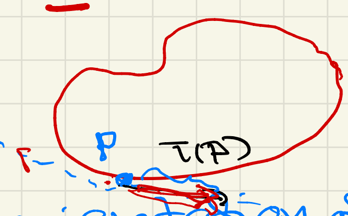
Now consider a closed oriented curve C. The circulation of the flow around C is the amount of the quantity Q that is transported around C per unit time.

1° At a point P on C vector field F pushes Q by the vector $F(P)$.



1° We are only interested in the component of $F(P)$ that pushes Q in the direction of C.

2° For this we consider at the tangent vector $T(P)$ at the curve C, which is:

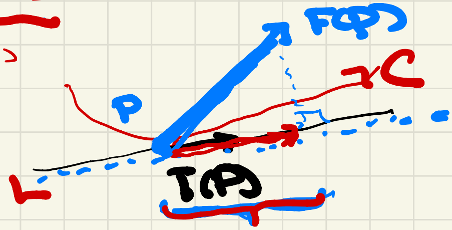


- of unit size and
- in the direction of the orientation of C.

3° The component of $F(P)$ in the direction of C, i.e., the direction of $T(P)$ is

$$F(P)\cdot T(P)$$

also called the projection $\mathrm{proj}_{T(P)} F(P)$ of $F(P)$ onto the line L given by $T(P)$.



$$\mathrm{proj}_{T(P)} F(P) = F(P)\cdot T(P).$$

4° This is amount of transportation of quantity Q that F does at the point P in the direction of the curve C

5° The circulation of Q along C is obtained by putting together those local contributions, i.e.

$$\text{circulation} = \int_C F\cdot T \, d\ell$$

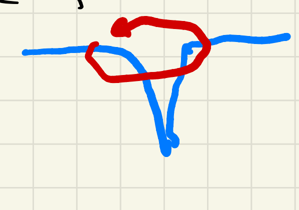
is the integral of the dot product over C and with respect to the length ℓ.

6° The integral of the tangent component of F is - we know - just the usual integral ("line integral") of F on the curve C. So:

$$\text{circulation} = \int_C F\cdot \underbrace{dr}_{=T\,d\ell}$$

This is our final formula for circulation.

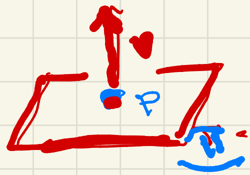
Ex.



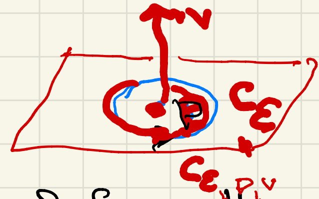
B. Circulation in the plane π orthogonal to a vector v

- **We** want to calculate circulation for the following curve.

Choose a Point P and a vector v, draw the plane π ("pi") through P and orthogonal to v:



- Now consider the circle C_ε in the plane π which has center at P, radius ε and orientation which is counter-clockwise when viewed from the tip of v.

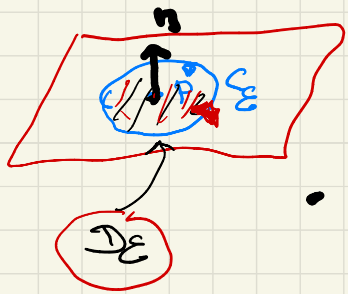


C. Use of Stokes theorem:

We want to calculate circulation $\boxed{\oint_{C_\varepsilon} F\cdot dr}$ around the circle C_ε using the Stokes theorem.

For this we need to choose an oriented surface S whose boundary is C_ε.

The obvious choice is the disk $\boxed{D_\varepsilon}$ with center P, radius ε, that lies in the plane π and has orientation n which is in the direction of v, i.e., $n = \frac{v}{|v|}$:



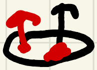
Then: $\boxed{\oint_{C_\varepsilon} F\cdot dr} = \oint_{\partial D_\varepsilon} F\cdot dr$

$C_\varepsilon = \partial D_\varepsilon$!

$\overset{\text{Stokes}}{=} \iint_{D_\varepsilon} \mathrm{curl}(F)\cdot d\vec{S}.$

E. Approximation of circulation

- We now know that the circulation around C_ε is

n 

$$\iint_{D_\varepsilon} \operatorname{curl}(F)\cdot d\vec{S} = \iint_{D_\varepsilon} \operatorname{curl}(F)\cdot n\, dA$$

where: n denotes the normal vectors on D_ε of unit length, which describe the orientation of the surface D_ε.

However; since D_ε lies in the plane Π, all normal vectors are the same. So:

$$n = n \text{ (the vector at } P) = \frac{v}{|v|}.$$

- So, circulation is

$$\iint_{D_\varepsilon} \operatorname{curl}(F)\cdot n \, dA.$$

- On a small disc D_ε it is reasonable to approximate the vector field $\operatorname{curl}(F)$ by its constant value at P:

we get:

$$[\text{circulation around } C_\varepsilon] \overset{\text{appr.}}{\approx} \iint_{D_\varepsilon} [\operatorname{curl}(F)](P)\cdot n \, dA$$

$$= [\operatorname{curl}(F)](P)\cdot n \iint_{D_\varepsilon} 1\, dA$$

$$\text{area} = A(D_\varepsilon) = \varepsilon^2\pi.$$

- So: the value at P

$$\boxed{n\cdot [\operatorname{curl}(F)](P)} \text{ is approximately } \frac{C_\varepsilon\text{-circulation}}{A(D_\varepsilon)}$$

F. The component curl(F) in the direction of n: $n \cdot \text{curl}(F)$ (calculate it exactly)

- In order to understand precisely the value of $n \cdot \text{curl}(F)$ at a point P, we need to $\boxed{\text{zoom-in}}$ in to P in the approximation:

$$[\text{curl}(F)](P) \cdot n \approx \frac{C_\varepsilon\text{-circulation}}{A(D_\varepsilon)}.$$

LHS RHS

P, D_ε

- This means that we should make ε smaller and smaller to capture what is happening very near to P and also at P.

As $\varepsilon \to 0$ we are considering approximation on smaller discs D_ε so approximation gets better (?)

In the limit as $\varepsilon \to 0$ the RHS approaches the exact value at P;

$$n \cdot [\text{curl}(F)](P) = \lim_{\varepsilon \to 0} \frac{C_\varepsilon\text{-circulation}}{\text{Area}(D_\varepsilon)}.$$

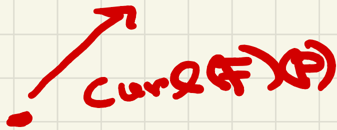
G. Conclusion:

Consider some P. For each unit vector n we can understand the component of curl(F)(P) in the n-direction:

this is the measure of circulation

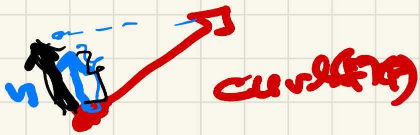
- around the point P
- in the plane orthogonal to n
- and calculated per unit area!

① We do not consider



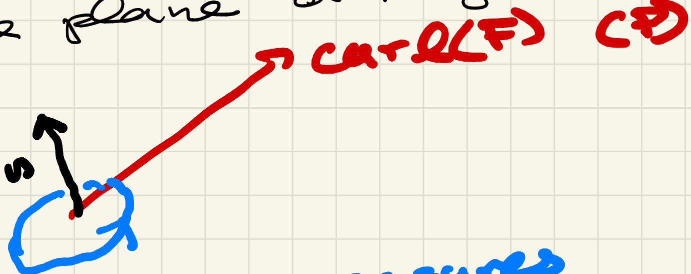
directly.

② We consider its **component** in a given direction $\underline{n}$: the black arrow



$= [\mathrm{curl}(F)](P) \cdot n$

③ This n component of $[\mathrm{curl}(F)](P)$ is the circulation of the v.f. F in the plane orthogonal to n:



④ curl(F)(P) measures the circulation of F in all planes π through P.

{Precisely: circulation per unit area near P.}

Weekend office hours

Saturday at 4 & Sunday

if needed!